# Options Portfolio Selection

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# Outline

- Problem:
  Optimal Investment in Options. Multiple Assets, Dependence.
- Model:
  One-Period Model. Infinitely Many Securities.
- Results:
  Optimal Portfolios and Performance.

# The Problem

- Options:
  Available on stocks, bonds, indices, futures, commodities.
  Usually available on dozens of strikes and a handful of maturities.
- S&P 500 index options returns: approximately -3% *a week*.
- Potentially high returns from selling options. Certainly high risks.
- How to construct optimal portfolios?
- High dimensional problem.
  Example: 10 assets $\times$ 20 strikes = 200 options. With a single maturity.
- Markowitz? Problematic.
  Options with only a small strike difference are nearly collinear.
  Nearly singular covariance matrix.

# One Asset

- With one asset and one maturity, problem tractable.
- $X$ underlying asset price at maturity.
  $c_X(K)$ price of a call option on $X$ with strike price $K$.
  $p_X(x)$ physical marginal density of $X$.
- Assume that continuum of strikes is available.
- Risk-neutral density $q_X(K)$ is (Breeden and Litzenberger, 1978)

$$q_X(K) := c_X''(K) \tag{1}$$

- Thus, the unique SDF is the random variable $m_X(x) = c_X''(x)/p_X(X)$.
- If the function $m_X$ is regular enough, the payoff decomposes as a portfolio of call and put options (Carr and Madan, 2001)

$$\begin{aligned} m_X(K) &= m_X(K_0) + m_X'(K_0)(K - K_0) \\ &\quad + \int_0^{K_0} m_X''(\kappa)(\kappa - K)^+ d\kappa + \int_{K_0}^{\infty} m_X''(\kappa)(K - \kappa)^+ d\kappa. \end{aligned}$$

- Payoffs with maximal Sharpe of the form $R = a + b\, m_X(X)$ with $b < 0$.

# Incompleteness with Multiple Assets

- Call and Put options available on all sorts of underlying assets.
- But each option depends only on one asset.
- Option prices identify risk-neutral marginals,
  but **not** the risk-neutral dependence structure.
- Infinitely many risk-neutral laws consistent with market marginals.
- Market incomplete.
- High dimensional problem, but not high enough to complete market...
- Which risk neutral law to use?
- It depends on the investor's objective.

# Literature

- Significant (negative) risk premia in options:
  Coval and Shumway (2001), Bakshi and Kapadia (2003), Santa-Clara and Saretto (2009), Schneider and Trojani (2015).
- Optimal payoff as weighted sum of calls and puts on all strikes.
  Carr and Madan (2001), Carr, Jin, Madan (2001).
- Performance manipulation with options on one asset: Goetzmann, Ingersoll, Spiegel, Welch (2007), Guasoni, Huberman, Wang (2011).
- Dynamic portfolio choice with options on one asset and one or two strikes: Liu and Pan (2003), Eraker (2013), Faias and Stanta Clara (2011).
- "Greek efficient" portfolios with multiple assets: Malamud (2014).

# The Model

- Simplifications: one maturity, continuum of strikes.
Shortest maturity options are most liquid. Strikes very numerous.
Over 200 for the S&P 500 index, over 100 for large stocks.
- One period. Underlying asset prices at end of period $X_1, \dots, X_n$.
Random variables on a probability space $(\Omega, \mathcal{F}, P)$, $\mathcal{F} = \sigma(X_1, \dots, X_n)$.
- By Carr-Madan formula, any smooth function $f$ of $X_i$ corresponds to a weighted average of options.
- Define options portfolio as a $n$-tuple $(f_1(x_1), \dots, f_n(x_n))$ of $L^2$ functions with finite price, defined as expecation under risk-neutral marginal.
- Optimal payoffs regular if densities regular.

# Portfolio Objective

- Assume zero safe rate to simplify notation.
- Payoff $Z = f_1(X_1) + \cdots + f_n(X_n)$ and price $\pi$.
- Maximize the Sharpe ratio, i.e., find the returns that

$$\max_R \frac{E[Z - \pi]}{\sigma(Z)}$$

- Payoff identified up to scaling and price.
  $Z$ optimal iff $a + bZ$ optimal, with $b > 0$.
- Ubiquitous objective in performance evaluation.
- And tractable.

# Duality

- Maximixing Sharpe ratio equivalent to minimizing variance of SDF.
- Convex $\mathcal{R} \subset L^2(\mathcal{F}, P)$ space of payoffs.
- Assume some SDF $\hat{M} > 0$ characterizes prices, and denote all SDFs by

$$\mathcal{M} = \{M \in L^2, \mathbb{E}[RM] = \mathbb{E}[R\hat{M}] \text{ for all } R \in \mathcal{R}\}.$$

- Implies that for any excess return:

$$0 = E[RM] = \text{cov}(R, M) + \mathbb{E}[R]\mathbb{E}[M] \geq -\sigma(R)\sigma(M) + \mathbb{E}[R]$$

- Whence Hansen-Jagannathan bound:

$$\sup_{\substack{R \in \mathcal{R} \\ \sigma(R) \neq 0, \mathbb{E}[MR]=0}} \frac{\mathbb{E}[R]}{\sigma(R)} \leq \inf_{M \in \mathcal{M}} \sigma(M)$$

- Morale: instead of looking for $R$, look for SDF $M^*$ with minimal variance.
- If $M^*$ is a payoff, $R = -M^* + E[(M^*)^2]$ spans all optimal returns.

# Dual Problem

- To ease notation: two assets with payoffs $X$ and $Y$. Solve

$$\min_{M \in \mathcal{M}} E[M^2]$$

subject to the restrictions

$$E[M|X] = \frac{q_X(X)}{p_X(X)}, \qquad E[M|Y] = \frac{q_Y(Y)}{p_Y(Y)}.$$

- To guess solution, consider SDF of the form $M = m(X, Y)$. (Intuitively, other sources of randomness would only increase variance.)
- Two families of infinitely many constraints: Lagrange multipliers?
- Reformulate problem in terms of densities.

# Densities

- Find $m(x,y)$ that minimizes (interval $(0,\infty)$ used for concreteness)

$$\int_0^\infty \int_0^\infty m(x,y)^2 p(x,y) dxdy$$

subject to the constraints

$$\int_0^\infty m(x,y)\frac{p(x,y)}{p_X(x)}dy = \frac{q_X(x)}{p_X(x)} \qquad \int_0^\infty m(x,y)\frac{p(x,y)}{p_Y(y)}dx = \frac{q_Y(y)}{p_Y(y)}$$

- Formally, rewrite as unconstrained problem:

$$\begin{aligned}\int_0^\infty \int_0^\infty m(x,y)^2 p(x,y) dxdy &- \int_0^\infty \Phi_X(x)\left(\int_0^\infty m(x,y)p(x,y)dy - q_X(x)\right)dx \\ &- \int_0^\infty \Phi_Y(y)\left(\int_0^\infty m(x,y)p(x,y)dx - q_Y(y)\right)dy,\end{aligned}$$

- Functions $\Phi_X(x)$ and $\Phi_Y(y)$ as infinite-dimensional Largrange multipliers.

# Integral Equations

- Eliminating constant terms, equivalent to:

$$\int_0^\infty \int_0^\infty (m(x,y) - \Phi_X(x) - \Phi_Y(y))\, m(x,y)p(x,y)dxdy.$$

- Setting first-order variation to zero leads to candidate solution

$$m^*(x,y) = \frac{1}{2}(\Phi_X(x) + \Phi_Y(y))$$

where $\Phi_X(x)$ and $\Phi_Y(y)$ are identified by the system of equations

$$\begin{aligned} \frac{1}{2}\Phi_X(x)p_X(x) + \frac{1}{2}\int_0^\infty \Phi_Y(y)p(x,y)dy =& q_X(x) \qquad x > 0, \\ \frac{1}{2}\int_0^\infty \Phi_X(x)p(x,y)dx + \frac{1}{2}\Phi_Y(y)p_Y(y) =& q_Y(y) \qquad y > 0. \end{aligned}$$

- Does this have a solution?
- If $(\Phi_X, \Phi_Y)$ works, then $\Phi'_X(x) = \Phi'_X(x) + c, \Phi'_Y(y) = \Phi_Y(y) - c$ also works.
- Eliminate degree of freedom by setting

$$\int_0^\infty \Phi_X(x)p_X(x)dx = \int_0^\infty \Phi_Y(y)p_Y(y)dy$$

# Main Result (1/2)

## Theorem

*Assume that* $\mathcal{M} \neq \emptyset$ *and* $\left\| \frac{p_i p_i^c}{p} \right\|_p^2 < \infty, 1 \leq i \leq n$. *Then:*

- *(Existence and Uniqueness) There exists a unique minimal SDF* $M^* \in \mathcal{M}$.
- *(Linearity) There exist* $\Phi := (\Phi_1, \ldots, \Phi_n)$, *where each* $\Phi_i \in L_p^2$ *for* $1 \leq i \leq n$, *such that the SDF is of the form* $M^* = m^*(X)$, *where*

$$m^*(\xi) = \frac{1}{n} \sum_{i=1}^n \Phi_i(\xi_i).$$

- *(Identification)* $\Phi$ *is the unique solution to the system of integral equations*

$$p_i(\xi_i)\Phi_i(\xi_i) + \sum_{j \neq i} \int_{\mathcal{D}_i^c} \Phi_j(\xi_j) p(\xi) d\xi_i^c = n q_i(\xi_i)$$

*with the uniqueness constraints* $\int_{I_i} \Phi_i(\xi_i) p_i(\xi_i) d\xi_i = 1, 1 \leq i \leq n$.

# Main Result (2/2)

## Theorem

- *(Performance) Optimal excess returns are of the form* $a(m^* - \mathbb{E}[(m^*)^2])$ *for* $a < 0$*, and their common maximum Sharpe ratio is*

$$SR = \sqrt{\frac{1}{n}\sum_{i=1}^{n}\int_{I_i}\Phi_i(\xi_i)q_i(\xi_i)d\xi_i - 1}. \quad (2)$$

- *(Regularity) Let* $(q_i)_{i=1}^n \subset C^k(\mathbb{R})$ *with* $k \geq 0$*. Denoting the continuous partial derivatives by* $\partial_{\xi_i}^{\beta} p(\xi)$, $0 \leq \beta \leq k$*, if for any* $R > 0$ *there exists* $\alpha \in (1/2, 1]$ *such that*

$$\sup_{\xi:\|\xi_i\|\leq R}\left|\frac{\partial_{\xi_i}^{\beta} p(\xi)}{(p_i^c(\xi_i^c))^{\alpha}}\right| < \infty \qquad \int_{\mathcal{D}_i^c}(p_i^c(\xi_i^c))^{2\alpha-1}d\xi_i^c < \infty,$$

*then* $m^*(\xi) = \frac{1}{n}\sum_{i=1}^{n}\Phi_i(\xi_i)$ *is also in* $C^k(\mathbb{R})$*.*

# Sanity Checks

- Risk-Neutrality:
  If options prices reflect zero risk premium $q_X/p_X = q_Y/p_Y = 1$, then we should neither buy nor sell them.
- Indeed, in this case $\Phi_X = \Phi_Y = 1$, whence $m^* = 1$, which has zero variance.
- Independence:
  If $X$ and $Y$ are independent under $p$, then the optimization problem should separate across assets.
- Indeed, $\Phi_X(x) = 2\frac{q_X(x)}{p_X(x)} - 1$, $\Phi_Y(y) = 2\frac{q_Y(y)}{p_Y(y)} - 1$. No interaction.
  $m^*(x,y) = \frac{q_X(x)}{p_X(x)} + \frac{q_Y(y)}{p_Y(y)} - 1$.
- Trivial example, nontrivial message.
  If options on multiple underlyings are not traded, the risk-neutral density consistent with independence and the maximization of the Sharpe ratio is $q_{X,Y}(x,y) = q_X(x)p_Y(y) + q_Y(y)p_X(x) - p_X(x)p_Y(y)$. It does not correspond to any particular copula...
- Nontrivial explicit solutions with dependence?
- Tractability?

# Mixture Distributions (1/2)

- Solving integral equations is nontrivial. To break the spell, discretize.
- $(p_X^i)_{1\leq i\leq k}$, $(p_Y^i)_{1\leq i\leq k}$ strictly positive probability densities on $(0,\infty)$.

$$p(x,y) := \frac{1}{k}\sum_{i=1}^{k} p_X^i(x)p_Y^i(y).$$

(Remember the proof of Fubini-Tonelli theorem?)

- Plug into integral equations. They become

$$\frac{p_X(x)}{2}\Phi_X(x) = q_X(x) - \sum_{i=1}^{k} c_Y^i p_X^i(x), \quad \frac{p_Y(y)}{2}\Phi_Y(y) = q_Y(y) - \sum_{i=1}^{k} c_X^i p_Y^i(y),$$

where the $2k$ constants $(c_X^i)_{1\leq i\leq k}, (c_Y^i)_{1\leq i\leq k}$ are

$$c_X^i = \frac{1}{2k}\int_0^\infty \Phi_X(x)p_X^i(x)dx, \quad c_Y^i = \frac{1}{2k}\int_0^\infty \Phi_Y(y)p_Y^i(y)dy.$$

- Plug formulas for $\Phi_X$ and $\Phi_Y$ again.

# Mixture Distributions (2/2)

- Obtain system of $2k$ equations in $2k$ unknowns

$$c_Y^i = \frac{1}{k}\int_0^\infty q_Y(y)\frac{p_Y^i(y)}{p_Y(y)}dy - \frac{1}{k}\sum_{j=1}^{k} c_X^j \int_0^\infty \frac{p_Y(y)^j p_Y^i(y)}{p_Y(y)}dy \qquad 1 \le i \le k$$

$$c_X^i = \frac{1}{k}\int_0^\infty q_X(x)\frac{p_X^i(x)}{p_X(x)}dx - \frac{1}{k}\sum_{j=1}^{n} c_Y^j \int_0^\infty \frac{p_X^j(x) p_X^i(x)}{p_X(x)}dx \qquad 1 \le i \le k.$$

- But the rank is $2k - 1$.
- Drop one equation and replace it with the uniqueness constraint

$$\sum_{i=1}^{k} c_X^i - \sum_{i=1}^{k} c_Y^i = 0.$$

- Now system is invertible.
- Note: $k$ in mixture representation independent of number of assets $n$. (Independence corresponds to a minimal $k = 1$ regardless of $n$.)
- No curse of dimensionality.

# Discrete Densities

- Another tractable discretization is with piecewise constant densities.
- Two increasing finite sequences $(x_i)_{0\leq i\leq k}$ and $(y_j)_{0\leq j\leq l}$.
- Assume $P(X\in[x_0,x_k), Y\in[y_0,y_l)) = Q(X\in[x_0,x_k), Y\in[y_0,y_l)) = 1$.
- Assume joint probability density $p$ constant on each rectangle $I_i^x\times I_j^y$, where $I_i^x=[x_{i-1},x_i)$, $1\leq i\leq k$, and $I_j^y=[y_{j-1},y_j)$, $1\leq j\leq l$.
- Denote $\tilde{p}^{ij}=P(X\in I_i^x, Y\in I_j^y)$, $\tilde{p}_X^i=P(X\in I_i^x)$, $\tilde{p}_Y^j=P(Y\in I_j^y)$, and $\tilde{q}_X^i=Q(X\in I_i^x)$, $\tilde{q}_Y^j=Q(Y\in I_j^y)$, $1\leq i\leq k, 1\leq j\leq l$.
- Any solution $\Phi_X,\Phi_Y$ piecewise constant on $(I_i^x)_{1\leq i\leq n}$ and $(I_j^y)_{1\leq j\leq m}$. Set $\Phi_X^i=\Phi_X(x_i)$ and $\Phi_Y^i=\Phi_Y(x_j)$.
- Integral equations reduce to:

$$\Phi_X^i\tilde{p}_X^i+\sum_{j=1}^{k}\Phi_Y^j\tilde{p}^{ij}=2\tilde{q}_X^i,\ 1\leq i\leq k, \Phi_Y^j\tilde{p}_Y^j+\sum_{i=1}^{l}\Phi_X^i\tilde{p}^{ij}=2\tilde{q}_Y^j,\ 1\leq j\leq l.$$

- Uniqueness constraint $\sum_{i=1}^{n}\Phi_X^i\tilde{p}_X^i-\sum_{j=1}^{m}\Phi_Y^j\tilde{p}_Y^j=0$.
- Curse of dimensionality.

# Example: Variance Gamma Model

- Common wisdom on option portfolios:
  Writing options profitable but risky. Diversify over many assets.
- Which strikes to write more? Impact of correlation?
- Example: Variance-Gamma model.
  Combines no-arbitrage with different realized and implied volatilities.
  Important to separate options' risk-premia from assets' risk premia.
- Two risky asset prices, both distributed as

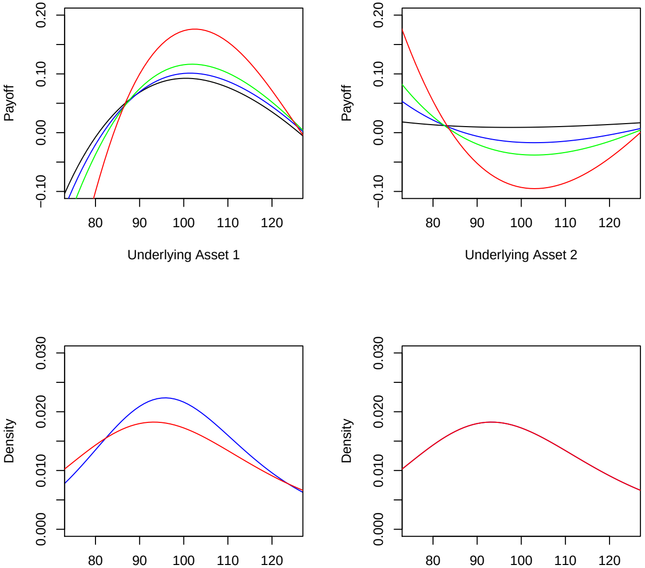
$$X_t = X_0 e^{\omega t + Z_t(\sigma,\nu,\theta)},$$

  where $Z_t$ has the characteristic function

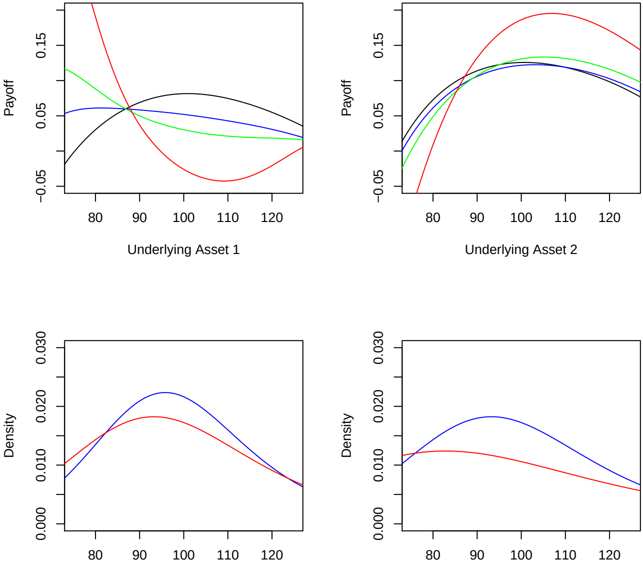
$$\mathbb{E}[e^{iuZ_t}] = (1 - i\theta\nu u + \frac{\sigma^2}{2}u^2\nu)^{-t/\nu}, \quad u \in \mathbb{R}$$

- Marginal of a Levy process with jump measure $k_Z(x) = \frac{e^{\theta x/\sigma^2}}{\nu|x|} e^{-\frac{\sqrt{\frac{2}{\nu}+\frac{\theta^2}{\sigma^2}}}{\sigma}|x|}$.
- Dependence modeled through bivariate $t$-copula.
- Assets' risk premia both zero.

# $\sigma_X^P = 20\%, \sigma_X^Q = \sigma_Y^Q = \sigma_Y^P = 25\%$



# $\sigma_X^P = 20\%, \sigma_X^Q = 25\%, \sigma_Y^P = 25\%, \sigma_Y^Q = 40\%$



# Performance

| Correlation | Figure 1 (annual) | Figure 1 (monthly) | Figure 2 (annual) | Figure 2 (monthly) |
|---|---|---|---|---|
| 0% | 0.29 | 0.68 | 0.62 | 1.71 |
| 60% | 0.31 | 0.74 | 0.58 | 1.63 |
| 75% | 0.33 | 0.84 | 0.58 | 1.67 |
| 90% | 0.43 | 1.17 | 0.63 | 1.99 |

- Annualized Sharpe ratios of optimal portfolios.
- Trade annually (left) or monthly (right).
- Higher correlation? Higher Sharpe ratio.
  Against intuition on diversification.
- Reason: correlation is among assets, not all options.
- Keeping the same marginals while increasing correlation increases the diversification and hedging opportunities among individual options.

# Conclusion

- Options portfolio selection.
- Each option on one underlying asset.
  Market incomplete with multiple assets.
- Maximize Sharpe ratio:
  system of linear integral equations.
- Integral equations intractable virtually all nontrivial cases.
  Discretizations tractable in virtually all cases.
- Optimal payoffs in one asset depend on options prices in all other assets.
  Except with independence.
- It may be optimal to buy options in one asset, expecting to lose.
  Just to hedge more profitable options in another asset.

# Thank You!

## Questions?

http://ssrn.com/abstract=3075945